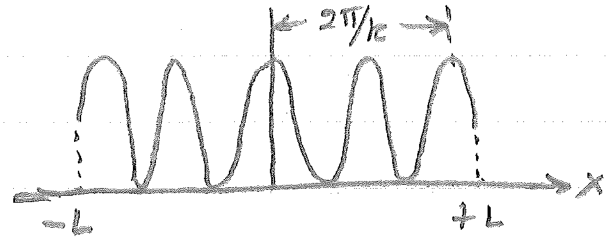
P302 Photonics Fall 2010

Solutions to HW # 10

11.2 If $f(x) = \begin{cases} \sin^2 k_p x \, , & |x| < L \\ 0 \, , & |x| > L \end{cases}$



The $F(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$

$$= \int_{-L}^{L} \sin^2 k_p x \, e^{ikx} dx = \int_{-L}^{L} \left(\frac{e^{ik_p x} - e^{-ik_p x}}{2i} \right)^2 e^{ikx} dx$$

$$= -\frac{1}{4} \int_{-L}^{L} \left(e^{i2k_p x} + e^{-i2k_p x} - 2 \right) e^{ikx} dx$$

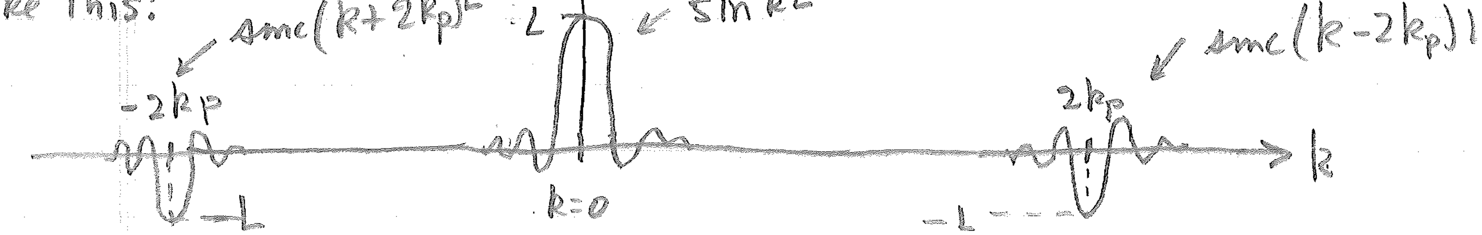
$$= -\frac{1}{4} \int_{-L}^{L} \left[e^{i(k+2k_p)x} + e^{i(k-2k_p)x} - 2e^{ikx} \right] dx$$

$$= -\frac{1}{4} \left\{ \left. \frac{e^{i(k+2k_p)x}}{i(k+k_p)} \right|_{-L}^{L} + \left. \frac{e^{i(k-2k_p)x}}{i(k-2k_p)} \right|_{-L}^{L} - \left. \frac{2e^{ikx}}{ik} \right|_{-L}^{L} \right\}$$

$$= -\frac{1}{4} \left\{ \frac{2\sin(k+2k_p)L}{(k+2k_p)} + \frac{2\sin(k-2k_p)L}{(k-2k_p)} - \frac{4\sin kL}{k} \right\}$$

$$\boxed{F(k) = L \,\mathrm{sinc}\, kL - \frac{L}{2} \mathrm{sinc}(k+2k_p)L - \frac{L}{2} \mathrm{sinc}(k-2k_p)L}$$

The plot of $F(k)$ depends on the value of k_p. If $2k_p > \pi/L$, it looks like this:



The half width of each central peak is π/L

(11.9) $F(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$

(a) Thus $\mathcal{F}\{f(\frac{x}{a})\} = \int_{-\infty}^{\infty} f(\frac{x}{a})\, e^{ikx}\, dx$

Now let $\frac{x}{a} = x'$, ie $x = ax'$, so $dx = a\,dx'$

$$\Rightarrow \mathcal{F}\{f(\tfrac{x}{a})\} = \int_{-\infty}^{\infty} f(x')\, e^{i(ak)x'} \cdot a\,dx'$$

$$\underline{\mathcal{F}\{f(\tfrac{x}{a})\} = a\,F(ak)}$$

(b) $\mathcal{F}\{f(-x)\} = \int_{-\infty}^{\infty} f(-x)\, e^{ikx}\, dx$

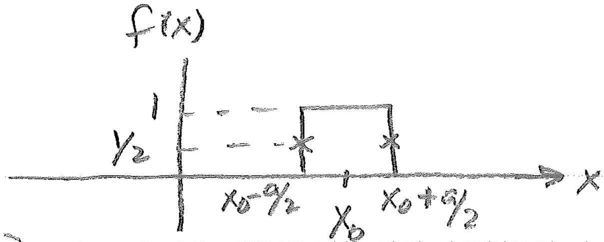
Let $-x = x'$, ie $dx = -dx'$. Also when $x \to \mp\infty \Rightarrow x' \to \pm\infty$

$$\text{so } \mathcal{F}\{f(-x)\} = -\int_{+\infty}^{-\infty} f(x')\, e^{i(-k)x'}\, dx'$$

$$= +\int_{-\infty}^{\infty} f(x')\, e^{i(-k)x'}\, dx'$$

$$\underline{\mathcal{F}\{f(-x)\} = F(-k)}$$

(11.11) $f(x) = \text{rect}\left|\frac{x-x_0}{a}\right|$

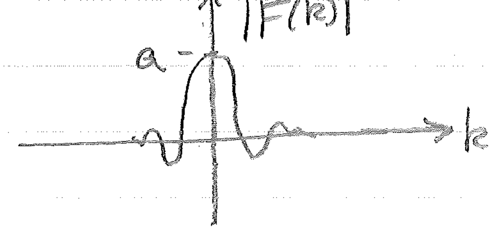


Then $F(k) = \mathcal{F}\left\{\text{rect}\left|\frac{x-x_0}{a}\right|\right\}$

$$= \int_{-\infty}^{\infty} \text{rect}\left|\frac{x-x_0}{a}\right| e^{ikx}\, dx$$

$$= \int_{x_0-a/2}^{x_0+a/2} e^{ikx}\, dx = \frac{e^{ik(x_0+\frac{a}{2})} - e^{ik(x_0-a/2)}}{ik}$$

$$\underline{F(k) = a e^{ikx_0} \operatorname{sinc}\frac{ka}{2}}$$



(11.12) From (11.11), with $x_0 = 0$ and $a = 1$

$$\operatorname{sinc}\frac{k}{2} = \mathcal{F}\{\text{rect}|x|\}$$

Taking another Fourier transform:

$$\mathcal{F}\left\{\operatorname{sinc}\frac{k}{2}\right\} = \mathcal{F}\{\mathcal{F}\{\text{rect}|x|\}\}$$

Using the results of (11.10), this is:

$$\mathcal{F}\left\{\operatorname{sinc}\frac{k}{2}\right\} = 2\pi\, \text{rect}\,(|-x|) = 2\pi\, \text{rect}\,|x|$$

or $\displaystyle\int_{-\infty}^{\infty} \operatorname{sinc}\frac{k}{2} e^{ikx}\, dk = 2\pi\, \text{rect}\,|x|$

(11.12) cont'd : Change variables,
Now let $x = k'$ and $k = x'$

$$\int_{-\infty}^{\infty} \text{sinc}\,\frac{x'}{2}\, e^{ik'x'}\,dx' = 2\pi\,\text{rect}\,|k'|$$

Now let $x' \to x$ and $k' \to k$

$$\int_{-\infty}^{\infty} \text{sinc}\,\frac{x}{2}\, e^{ikx}\,dx = 2\pi\,\text{rect}\,|k|$$

or $\mathcal{F}\left\{\frac{1}{2\pi}\,\text{sinc}\,\frac{x}{2}\right\} = \text{rect}\,|k|$

(11.20) Let $h(x) = \delta(x)$. Consider the convolution integral

$$g(X) = f \circledast h = \int_{-\infty}^{\infty} f(x)\,h(X-x)\,dx$$

ie $g(X) = \int_{-\infty}^{\infty} f(x)\,\delta(X-x)\,dx$

Since "δ" is an even function, $\delta(X-x) = \delta(x-X)$

so $g(X) = \int_{-\infty}^{\infty} f(x)\,\delta(x-X)\,dx$

$$\underline{g(X) = f(X)}$$

ie $f \circledast \delta = f(X)$